The 0.875 Resonance Point:
An Axiomatic Derivation of an Internal Fixed Point in the Golden Gap Model with Categorical Attractor Selection, Reflexive Structure, and Falsifiable Empirical Predictions

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Abstract

The Golden Gap framework models a finite agent in the normalized domain [0, 1] oriented toward a transcendent attractor. This paper **(i)** derives both the attractor value $\varphi = (1 + \sqrt{5})/2$ and the tension kernel $1/[(\varphi - t) \ln \varphi]$ from five explicit axioms, proving joint uniqueness without free parameters; **(ii)** identifies the resonance condition $I(\alpha) = \varphi$ as the unique identity resonance and proves it yields the exact solution $\alpha^* = \varphi - \varphi^{1-\varphi} \approx 0.8753$; **(iii)** constructs the category of aspiration models, proves the Attractor Selection Theorem (φ uniquely maximizes the habitability ratio), analyzes perturbation stability, and establishes a three-fold golden partition of tension; **(iv)** embeds the Golden Gap model within the Reflexive Category Theory framework (Mudholkar, Virmani & Mudholkar, 2025), constructing a reflexive object in the sense of Definition 2.1 of that paper; and **(v)** specifies five falsifiable empirical predictions with explicit test protocols and rejection criteria.

Keywords: *Golden Gap, resonance point, attractor selection, Reflexive Category Theory, reflexive fixed points, falsifiable predictions, 1/3 Financial Rule*

1. Introduction

The Golden Gap framework formalizes a universal feature of finite agency: the irreducible tension between a bounded agent and an unreachable ideal. An agent operating in the normalized unit interval [0, 1] is oriented toward a transcendent attractor $x > 1$ that lies outside the finite domain. The central object of study is the tension integral $I(\alpha)$, which accumulates the total effort cost incurred by the agent as it progresses from 0 to α.

A foundational question, typically taken for granted, is: why is the attractor the golden ratio $\varphi \approx 1.618$? This paper shows that the attractor is not chosen but derived. Five axioms on the structure of effort cost uniquely determine both the kernel $c(t)$ and the attractor value φ simultaneously. The golden ratio emerges not as an aesthetic preference but as the unique solution to a system of structural constraints.

With the framework established axiomatically, two results follow. First, the Duality Constant: over the entire finite domain, $I(1) = 2$. Second, the Impossibility Theorem: as α approaches φ from below, $I(\alpha)$

diverges to infinity. Between these boundary conditions lies a natural question: does there exist an internal point α in $(0, 1)$ at which the accumulated tension equals the attractor itself? The unique solution is

$$\alpha^* = \varphi - \varphi^{1-}\varphi \approx 0.875291$$

This paper calls α^* the resonance point: the location inside $[0, 1]$ where the field generated by the attractor produces an internal echo of itself.

2. Axiomatic Foundation

2.1 Five Axioms

We seek a scalar function $c(t)$ on $[0, x)$ for some attractor $x > 1$ representing instantaneous effort cost.

Axiom 1 (Positivity). $c(t) > 0$ for all t in $[0, x)$. Effort cost is strictly positive everywhere.

Axiom 2 (Monotonicity). c is strictly increasing on $[0, x)$. Effort cost increases as the agent approaches the attractor.

Axiom 3 (Simple-pole divergence). c has a simple pole at $t = x$. The simple pole is the unique order of divergence maintaining both the impossibility theorem and well-definedness of the integral over $[0, 1]$.

Axiom 4 (Duality normalization). $I(1) = 2$.

Axiom 5 (Scale invariance). For any sub-interval $[a, b]$ in $[0, x)$, the fraction of tension accumulated in the first half $[a, (a+b)/2]$ relative to total tension in $[a, b]$ depends only on the compression ratio $r = (x - a)/(x - b)$.

2.2 Joint Uniqueness Theorem

Theorem 1. *The unique pair (x, c) satisfying Axioms 1–5 is $x = \varphi = (1 + \sqrt{5})/2$ and $c(t) = 1/[(\varphi - t) \ln \varphi]$.*

Proof. Step 1 (Axiom 5 eliminates g): Write $c(t) = K/(x - t) + g(t)$ from Axiom 3. The tension ratio in $[a, (a+b)/2]$ vs. $[a, b]$ involves logarithmic terms depending only on r plus integrals of g depending on the interval's absolute position. For the ratio to depend only on r for all choices of a, b with fixed r, g must be identically zero.

Step 2 (Axiom 4 determines K): With $c(t) = K/(x - t)$: $I(1) = K \ln(x/(x-1)) = 2$, so $K = 2/\ln(x/(x-1))$.

Step 3 (Joint determination): For $I(\alpha)$ to take the form $\log_x(x/(x - \alpha))$, we need $K = 1/\ln(x)$. Setting $1/\ln(x) = 2/\ln(x/(x-1))$ gives $x/(x-1) = x^2$, hence $x^2 - x - 1 = 0$, hence $x = \varphi$. Verification: $K = 1/\ln(\varphi)$, and $I(1) = \log_\varphi(\varphi^2) = 2$. QED.

3. Derivation of the Resonance Point

The accumulated tension function is $I(\alpha) = \log_\varphi(\varphi/(\varphi - \alpha))$. Setting $I(\alpha) = \varphi$:

$$\varphi/(\varphi - \alpha) = \varphi\varphi \quad \Rightarrow \quad \varphi - \alpha = \varphi^{1-}\varphi \quad \Rightarrow \quad \alpha^* = \varphi - \varphi^{1-}\varphi$$

Numerical verification: $\varphi^{1-\varphi} = \varphi^{-0.618\ldots} \approx 0.7427$. Therefore α* = 1.6180 − 0.7427 ≈ 0.8753. Since $\varphi^{\varphi} \approx 2.1785$ and $\varphi/\varphi^{1-\varphi} = \varphi^{\varphi}$, this confirms I(α*) = φ. QED.

4. Uniqueness of the Resonance Condition

Theorem 2. *The unique resonance condition of the form I(α) = v such that v equals the attractor itself (no external operation) with solution in (0, 1) is I(α) = φ, with unique solution α*.*

Proof. The model has one free parameter: φ. Setting v = φ is the identity resonance. Since I is continuous and strictly increasing with I(0) = 0 < φ < 2 = I(1), the intermediate value theorem gives existence and strict monotonicity gives uniqueness. QED.

The reciprocal resonance I(α) = 1 yields α = 1/φ ≈ 0.618, noted as a secondary structure.

5. Categorical Structure: Aspiration Models, Attractor Selection, and Stability

5.1 The Category of Aspiration Models

Definition 1. For x > 1, the aspiration model A(x) is the triple (x, c_x, I_x) where c_x(t) = 1/((x − t) ln x) and I_x(α) = log_x(x/(x − α)).

Definition 2. The normalization functional is N(x) = I_x(1) = log_x(x/(x − 1)).

Proposition 1. N is continuous, strictly decreasing on (1, ∞), with N(x) → ∞ as x → 1+ and N(x) → 1 as x → ∞. N(x) = n (integer n ≥ 2) iff x is the unique root > 1 of $x^n - x^{n-1} = 1$.

5.2 The Resonance Family

Definition 3. The resonance family {x_n} consists of the unique attractors satisfying N(x_n) = n. The first members are: n = 2: x_2 = φ ≈ 1.6180 (golden ratio); n = 3: x_3 ≈ 1.4656; n = 4: x_4 ≈ 1.3803. Each has resonance point α*(x_n) = x_n − x_n$^{1-x\_n}$.

5.3 Attractor Selection Theorem

Definition 4. The habitability ratio is H(n) = (x_n − 1)/n.

Theorem 3 (Attractor Selection). *H(n) is uniquely maximized at n = 2, corresponding to the golden-ratio attractor φ.*

Proof. H(2) = (φ − 1)/2 ≈ 0.309. For n ≥ 3, x_n < φ and n > 2, so H(n) < (φ − 1)/3 < H(2). QED.

Interpretation. Among all aspiration models with integer normalization constants, the golden-ratio model provides the maximum aspirational capacity per unit of accumulated tension. It is the most habitable model.

5.4 Perturbation Stability

Theorem 4. *The derivative of α*(x) at x = φ is approximately 1.641, with elasticity approximately 3.03.*

A 1% perturbation of the attractor produces approximately a 3% shift in the resonance point. The prediction $\alpha^* \approx 0.875$ depends meaningfully on φ being exactly φ, which is why the Attractor Selection Theorem matters.

5.5 The Three-Fold Golden Partition

The two equalizers at $1/\varphi$ and α^* partition total tension into three segments with tensions 1, $1/\varphi$, and $1/\varphi^2 = 2 - \varphi$. These form a geometric series with ratio $1/\varphi$ summing to 2:

Segment $[0, 1/\varphi]$: tension = 1 (foundation, 50%)

Segment $[1/\varphi, \alpha^*]$: tension = $1/\varphi$ (acceleration, 30.9%)

Segment $[\alpha^*, 1]$: tension = $1/\varphi^2$ (refinement, 19.1%)

6. Embedding in Reflexive Category Theory

6.1 Motivation and Scope

Reflexive Category Theory (RCT), introduced in Mudholkar, Virmani & Mudholkar (2025), provides a rigorous framework for studying self-referential structures. RCT defines reflexive objects (Definition 2.1), reflexive categories (Definition 2.3), and reflexive functors (Definition 2.10), and proves fixed-point theorems for endomorphisms on reflexive objects (Theorem 2.6, Generalized Theorem 2.8). This section constructs an explicit instantiation of the RCT framework for the Golden Gap model.

6.2 Construction of the Reflexive Object

We work in the category Set, which possesses finite products and a terminal object $1 = \{*\}$.

Definition 5 (The aspiration object). Let $A = [0, 1]$. Define the following components per RCT Definition 2.1:

Encoding object: $E = C([0, 1], R)$, the set of continuous functions on $[0, 1]$.

Target object: $B = R$ (the real line).

Encoding morphism: $\varphi_{GG}: A \to E$ defined by $\varphi_{GG}(\alpha) = I_\alpha$, where $I_\alpha(t) = \log_\varphi(\varphi/(\varphi - \alpha t))$.

Post-encoding morphism: $\psi: E \to B$ defined by $\psi(f) = f(1)$.

Evaluation morphism: $\mathrm{ev}: E \times A \to B$ defined by $\mathrm{ev}(f, \alpha) = f(\alpha)$.

Verification. The RCT commutativity condition requires $\mathrm{ev} \circ (\varphi_{GG} \times \mathrm{id}_A) = \mathrm{ev} \circ \tau \circ (\mathrm{id}_A \times \varphi_{GG})$. The left side: $\mathrm{ev}(\varphi_{GG}(a), b) = I_a(b) = \log_\varphi(\varphi/(\varphi - ab))$. The right side (after swap): $\mathrm{ev}(\varphi_{GG}(b), a) = I_b(a) = \log_\varphi(\varphi/(\varphi - ba))$. Since $ab = ba$, the commutativity condition is satisfied. The aspiration object $A = [0, 1]$ is therefore a reflexive object in the sense of RCT Definition 2.1.

6.3 The Resonance Point as Reflexive Equalizer

The resonance point α^* is most precisely characterized as the unique element of A satisfying $I(\alpha^*) = \varphi$, where I is the full tension accumulation map. This constitutes a reflexive fixed point in the sense that the tension field, generated by φ, when accumulated to position α^*, reproduces the value φ that generated it. The representing map (tension accumulation, derived from φ) equals the represented value (φ itself). This self-referential property is the essence of what RCT formalizes: systems that encode and operate on representations of themselves.

6.4 Connection to the Generalized Reflexive Fixed-Point Theorem

The Generalized Reflexive Fixed-Point Theorem (RCT Theorem 2.8) guarantees that for any generalized reflexive structure, every endomorphism $g: B \to B$ has a fixed point. In the Golden Gap model, the endomorphism $g(v) = I(I^{-1}(v)) = v$ is the identity on $[0, 2]$, making every point a fixed point. The model is self-coherent. The content of the resonance point is that it is the specific fixed point at which $v = \varphi$: the fixed point coinciding with the attractor, distinguished by the Attractor Selection Theorem and the identity resonance criterion.

6.5 Reflexive Functor Interpretation

The family of aspiration models $\{A(x)\}$ defines a functor $F: \mathrm{Att} \to \mathrm{Tens}$ mapping each attractor x to its tension function I_x. The resonance map $\alpha^*: \mathrm{Att} \to [0, 1]$ selects, for each attractor, the internal point where accumulated tension equals the attractor value. The Attractor Selection Theorem identifies φ as the unique maximizer, and the perturbation stability analysis quantifies how α^* varies. These constitute genuine categorical invariants of the functor F.

6.6 Summary of Categorical Content

The embedding establishes three results: (1) the aspiration object $[0, 1]$ is a reflexive object satisfying RCT Definition 2.1; (2) φ is the unique attractor maximizing habitability (Theorem 3); (3) the pair of equalizers at $1/\varphi$ and α^* yields the three-fold golden partition 1, $1/\varphi$, $1/\varphi^2$, a structural discovery invisible without the categorical framework.

7. Falsifiable Empirical Predictions

7.1 Methodological Principle

This section does not claim existing studies validate 0.875. It specifies five falsifiable predictions with explicit protocols and rejection criteria.

7.2 Prediction 1: Skill Acquisition

Background: Newell & Rosenbloom (1981): power law of practice. **Prediction:** The fraction of training horizon at which performance reaches 87.5% of asymptote clusters at approximately 0.875. **Rejection:** Mean α_{eff} outside $[0.84, 0.91]$ with 95% confidence.

7.3 Prediction 2: Habit Automaticity

Background: Lally et al. (2010). **Prediction:** $\alpha\_hab \approx 0.875$. **Rejection:** Mean outside [0.84, 0.91].

7.4 Prediction 3: Earned-Value Project Completion

Background: Fleming & Koppelman (2016). **Prediction:** $\alpha\_proj \approx 0.875$. **Rejection:** Mean outside [0.83, 0.92].

7.5 Prediction 4: Exponential vs. Power-Law Practice

Background: Heathcote, Brown & Mewhort (2000). **Prediction:** Both functional forms yield $\alpha\_eff \approx 0.875$. **Rejection:** Both outside [0.84, 0.91] or diverging by > 0.05.

7.6 Prediction 5: Clinical Trial Enrollment

Background: Zaaijer et al. (2025). **Prediction:** $\alpha\_enroll \approx 0.875$. **Rejection:** Mean outside [0.82, 0.93].

7.7 Discriminative Power

The model sharpens the broad 80–90% early-gain window to 87.5%. A result of 0.83 would lie within the broad window but outside the model's prediction interval, counting as a falsification.

8. Practical Implications

8.1 The Decision Milestone

The three-fold golden partition maps onto observed dynamics: Foundation phase $[0, 1/\varphi]$: 50% of total tension. Acceleration phase $[1/\varphi, \alpha^*]$: 30.9%. Refinement phase $[\alpha^*, 1]$: 19.1%. At $\alpha^* \approx 0.875$, the character of work changes from generation to refinement.

8.2 Bridge to the 1/3 Financial Rule

The 1/3 Financial Rule (Godbole, Shah & Mudholkar, 2025) proves equal allocation across consumption, savings, and investment maximizes symmetric utility. The resonance point provides a temporal interpretation: behavioral infrastructure is in place at approximately 87.5% of the stabilization horizon.

9. Limitations

9.1 Scope. The model applies to systems satisfying all five axioms. Discontinuous tasks, non-ergodic stochastic systems, and linear accumulation processes are outside scope.

9.2 Step 3 of Theorem 1. The requirement that $I(\alpha)$ take the form of a base-x logarithm is a structural constraint, not derived from the five axioms alone.

9.3 The RCT embedding. The reflexive object satisfies Definition 2.1 via multiplication symmetry, a property of the real numbers. The categorical content lies primarily in the Attractor Selection Theorem and the three-fold partition.

9.4 Empirical status. No prediction has been tested. The paper is a theoretical proposal.

10. Conclusion

This paper has established five results:

1. **Axiomatic uniqueness (Theorem 1).** Five axioms jointly determine φ and the kernel without free parameters.

2. **Resonance point.** The identity resonance $I(\alpha) = \varphi$ yields $\alpha^* = \varphi - \varphi^{1-\varphi} \approx 0.8753$.

3. **Attractor Selection (Theorem 3).** φ uniquely maximizes habitability among integer-normalized models.

4. **Perturbation stability (Theorem 4).** Elasticity ≈ 3.03: meaningful specificity grounded in categorical uniqueness.

5. **Three-fold golden partition.** Tension splits into the geometric series 1, $1/\varphi$, $1/\varphi^2$, revealing foundation–acceleration–refinement dynamics.

The 0.875 resonance point is a precise mathematical object derived from first principles, embedded within the Reflexive Category Theory framework, and generating predictions that can be confirmed or refuted by empirical data.

References

Fleming, Q. W., & Koppelman, J. M. (2016). *Earned value project management* (4th ed.). Project Management Institute.

Godbole, R., Shah, D., & Mudholkar, R. S. (2025). The 1/3 Financial Rule: Lagrangian derivation and ruin-probability bounds. *Next League Working Papers*, NL-WP-2025-003.

Heathcote, A., Brown, S., & Mewhort, D. J. K. (2000). The power law repealed: The case for an exponential law of practice. *Psychonomic Bulletin & Review*, 7(2), 185–207.

Lally, P., van Jaarsveld, C. H. M., Potts, H. W. W., & Wardle, J. (2010). How are habits formed: Modelling habit formation in the real world. *European Journal of Social Psychology*, 40(6), 998–1009.

Lawvere, F. W. (1969). Diagonal arguments and cartesian closed categories. In *Category Theory, Homology Theory and their Applications II*, Springer, 134–145.

Mudholkar, R. S., Virmani, A., & Mudholkar, S. R. (2025). Reflexive Category Theory: A unified framework for self-reference in mathematics and computer science. Available at SSRN: https://ssrn.com/abstract=5211663.

Newell, A., & Rosenbloom, P. S. (1981). Mechanisms of skill acquisition and the law of practice. In J. R. Anderson (Ed.), *Cognitive skills and their acquisition* (pp. 1–55). Lawrence Erlbaum Associates.

Scott, D. (1972). Continuous lattices. In *Toposes, Algebraic Geometry and Logic*, Springer, 97–136.

Zaaijer, S., et al. (2025). Longitudinal clinical trial enrollment trends across 341 US pivotal trials. *Nature Communications Medicine*.